

Fermionic quasiparticle representation of Tomonaga-Luttinger Hamiltonian

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Abstract. We find a unitary operator which asymptotically diagonalizes the Tomonaga-Luttinger Hamiltonian of one-dimensional spinless electrons. The operator performs a Bogoliubov rotation in the space of electron-hole pairs. If bare interaction of the physical electrons is sufficiently small this transformation maps the original Tomonaga-Luttinger system on a system of free fermionic quasiparticles. Our representation is useful when the electron dispersion deviates from linear form. For such situation we obtain non-perturbative results for the electron gas free energy and the density-density propagator.

PACS. 71.10.Pm Fermions in reduced dimensions (anyons, composite fermions, Luttinger liquid, etc.)

1 Introduction

Bosonization is a standard approach to the problem of interacting electrons in one dimension (1D) [1,2]. Bosonization maps the low-energy spectrum of the Tomonaga-Luttinger (TL) model on the spectrum of free bosons.

In this paper we discuss a new method of treating such system. We explicitly construct a unitary operator U diagonalizing TL Hamiltonian. Our unitary transformation maps the TL model on a system of free fermionic quasiparticles. The description of the TL model in terms of these quasiparticles has several advantages over bosonization, as we will see below.

The diagonalizing procedure is closely related to the bosonization. The operator U can be thought of as a Bogoliubov transformation in the space of particle-hole pair excitations. Alternatively, one can describe the action of U as a sequence of bosonization, Bogoliubov transformation which diagonalizes the bosonic Hamiltonian and further fermionization (Fig. 1).

When the dispersion of the physical electrons is linear both the quasiparticles and the bosons offer equally good description of the TL model. If the non-linear terms are substantial the free boson representation breaks down. The fermionic quasiparticles show more resilience toward deviations from the linear dispersion. They remain free as long as the bare interaction constant is sufficiently small.

This latter property of the quasiparticle representation allows for non-perturbative calculation of the free energy and the density-density correlation function for the

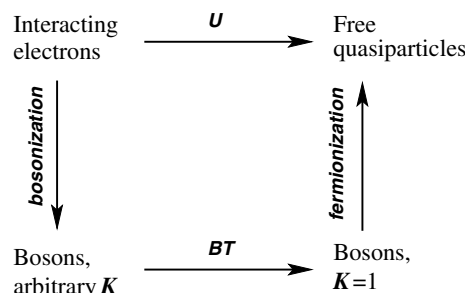


Fig. 1. Commutative diagram explains the relation between transformation U , and the bosonization. ‘BT’ stands for the Bogoliubov transformation of bosons.

TL model with the non-linear dispersion. We believe that these two results are new.

When the bare electron dispersion is linear the quasiparticle representation could be used to determine the single-electron Green’s function. It coincides with the Green’s function obtained by other methods.

The existence of the quasiparticles does not contradict to the fact that the single-electron Green’s function of TL model has no pole. The quasiparticles in TL model has zero overlap with the physical electrons: $\sqrt{Z} = 0$. Therefore, it is convenient to think about TL model as $Z = 0$ Fermi liquid [3].

The paper is organized as follows. We diagonalize the TL Hamiltonian in Section 2. In Section 3 we offer an intuitive explanation to the method. In this section all technical complications are disregarded in order to create an integral view. The density-density propagator is derived in Section 4. The single-particle Green’s function is

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calculated in Section 5. Section 6 is reserved for discussion. Certain technical details can be found in Appendices.

2 Diagonalization of TL Hamiltonian

The TL model Hamiltonian is given by:

$$H = H_{\text{kin}} + H_{\text{int}}, \quad (1)$$

$$H_{\text{kin}} = iv_F \int_{-L/2}^{L/2} dx \left(:\psi_L^\dagger \nabla \psi_L: - :\psi_R^\dagger \nabla \psi_R: \right), \quad (2)$$

$$H_{\text{int}} = \int dx dx' \hat{g}(x - x') \rho_L(x) \rho_R(x'), \quad (3)$$

$$\rho_{L,R} = :\psi_{L,R}^\dagger \psi_{L,R}:. \quad (4)$$

The chirality label ‘L’ corresponds to left-moving electrons, the label ‘R’ corresponds to right-moving electrons. The interaction of the electrons of the same chirality is ignored since up to irrelevant operators such interaction simply renormalizes the value of the Fermi velocity v_F . The symbol $:\dots:$ denotes normal ordering of the field operators ψ . A brief discussion of the normal ordering procedure we use in this paper is given in Appendix A.

It is assumed that the cut-off of (1) is infinite. To remove ultraviolet divergences of the theory without cut-off we replace usual zero-range interaction ($\hat{g}(x) = g_0 \delta(x)$) by interaction acting over a finite range. Specifically, $\hat{g}(x) = g_0 \delta_\Lambda(x)$ where $\delta_\Lambda(x)$ is a broadened version of the delta-function: its Fourier transform $\delta_\Lambda(q)$ is such that $\delta_\Lambda(q) = 1$ for $|q|$ smaller than some quantity Λ , and $\delta_\Lambda(q)$ vanishes quickly for $|q| > \Lambda$. The parameter Λ thus defined regularizes ultraviolet divergences of our theory.

It is easy to demonstrate (see, for example, [1]) that the following commutation relations are obeyed:

$$[\rho_{pq}, \rho_{p'-q'}] = \delta_{pp'} \delta_{qq'} p n_q \quad (5)$$

$$[\rho_{pq}, \psi_{p'}^\dagger(x)] = \delta_{pp'} e^{-iqx} \psi_{p'}^\dagger(x), \quad (6)$$

where $p = +1$ for left-moving electrons, $p = -1$ for right-moving electrons and

$$\rho_{pq} = \int dx e^{-iqx} \rho_p(x), \quad n_q = \frac{Lq}{2\pi}. \quad (7)$$

Using this commutation relations we will show that the unitary operator

$$U = e^\Omega, \quad (8)$$

$$\Omega = \sum_{q \neq 0} \sum_p \alpha_q \frac{p}{n_q} \rho_{p-q} \rho_{-pq} \quad (9)$$

diagonalizes the Hamiltonian (1) except for the zero mode part. Fortunately, the zero modes

$$N_p = \rho_{pq} \Big|_{q=0} \quad (10)$$

are decoupled from other degrees of freedom. Also, their contribution to the low-energy spectrum is $\mathcal{O}(1/L)$.

Since ρ 's are quadratic in ψ the above operator U is quartic in fermionic operators ψ . In general, it is impossible to work with such a non-linear object. In our situation, however, the simplicity of commutation rules (5) and (6) allows us to diagonalize the Hamiltonian with the help of U .

In order to transform the interaction term (3) with the operator U it is enough to observe that the action of U on the density operator ρ_{pq} , $q \neq 0$, is a Bogoliubov rotation:

$$U \rho_{pq} U^\dagger = u(\alpha_q) \rho_{pq} + v(\alpha_q) \rho_{-pq}, \quad (11)$$

$$u(\alpha_q) = u_q = \cosh(\alpha_q), \quad v(\alpha_q) = v_q = \sinh(\alpha_q), \quad (12)$$

$$u_q^2 - v_q^2 = 1. \quad (13)$$

This result is a simple consequence of the commutation relation (5). This identity can be derived by a variety of methods. For example, one can differentiate its left-hand side with respect to α_q for both $p = L$ and $p = R$ and solve the resultant differential equation system.

The easiest way to transform the kinetic energy term is to notice that the kinetic energy density can be expressed as a product of two density operators. The derivation goes as follows. First, we normal order the product of two density operators:

$$\begin{aligned} \rho_p(x) \rho_p(y) &= :\psi_p^\dagger(x) \psi_p(x) \psi_p^\dagger(y) \psi_p(y): \\ &+ s_p(x-y) : \psi_p(x) \psi_p^\dagger(y) : \\ &+ s_p(x-y) : \psi_p^\dagger(x) \psi_p(y) : + b_p(x-y), \end{aligned} \quad (14)$$

$$s_p(x) = \frac{p}{2\pi i (x - ip0)}, \quad (15)$$

$$b_p(x) = (s_p(x))^2. \quad (16)$$

As it is explained in Appendix A the normal ordering is used here to isolate explicitly singular terms of the field operator products. Now we expand the above identity into Laurent series with respect to powers of $(x - y)$:

$$\begin{aligned} \rho_p(x) \rho_p(y) &= b_p(x-y) + \frac{ip}{2\pi} : \psi_p(x) \nabla \psi_p^\dagger(x) : \\ &+ \frac{ip}{2\pi} : \psi_p^\dagger(x) \nabla \psi_p(x) : + (\text{irrelevant operators}). \end{aligned} \quad (17)$$

In this expansion an irrelevant operator can be recognized by a factor of $(x - y)^n$ where $n > 0$. For example, the first term in (14) is an irrelevant operator:

$$\begin{aligned} &:\psi_p^\dagger(x) \psi_p(x) \psi_p^\dagger(y) \psi_p(y): \\ &\approx (x - y)^2 : \psi_p^\dagger(x) \psi_p(x) \nabla \psi_p^\dagger(x) \nabla \psi_p(x) :. \end{aligned} \quad (18)$$

Indeed, by simple power counting one can verify that its scaling dimension is equal to $d = 4 < 2$. Sending $y \rightarrow x$ in equation (17), we establish the identity:

$$\begin{aligned} &\frac{ip}{2\pi} (: \psi_p^\dagger(x) (\nabla \psi_p(x)) : - : (\nabla \psi_p^\dagger(x)) \psi_p(x) :) \\ &= \lim_{y \rightarrow x} \{ \rho_p(x) \rho_p(y) - b_p(x-y) \}. \end{aligned} \quad (19)$$

Thus, it is permissible to write the Hamiltonian (1) in the form:

$$H = \frac{\pi v_F}{L} \sum_{pq} \left(\rho_{pq} \rho_{p-q} + \frac{g_q}{2\pi v_F} \rho_{pq} \rho_{-p-q} \right), \quad (20)$$

where g_q is the Fourier transform of $\hat{g}(x)$. For convenience we explicitly show the zero mode part of the above expression:

$$H = \frac{\pi v_F}{L} \sum_{q \neq 0} \sum_p \left(\rho_{pq} \rho_{p-q} + \frac{g_q}{2\pi v_F} \rho_{pq} \rho_{-p-q} \right) + \frac{\pi v_F}{L} (N_L^2 + N_R^2) + \frac{g_0}{L} N_L N_R. \quad (21)$$

Such splitting is useful since our transformation U does not act on the zero modes. We now apply U and choose parameters α_q in such a way that the term $\rho_L \rho_R$ vanishes:

$$\tanh 2\alpha_q = -\frac{g_q}{2\pi v_F}, \quad (22)$$

$$u_q^2 = \frac{1}{2} \left(1 + \frac{1}{\sqrt{1 - (g_q/2\pi v_F)^2}} \right),$$

$$v_q^2 = \frac{1}{2} \left(-1 + \frac{1}{\sqrt{1 - (g_q/2\pi v_F)^2}} \right), \quad v_q g_q < 0. \quad (23)$$

The transformed Hamiltonian is:

$$UHU^\dagger = \frac{\pi v_F}{L} \sum_{q \neq 0} \sum_p \left(u_q^2 + v_q^2 + \frac{g_q}{\pi v_F} u_q v_q \right) \rho_{pq} \rho_{p-q} + \frac{\pi v_F}{L} (N_L^2 + N_R^2) + \frac{g_0}{L} N_L N_R \quad (24)$$

$$= \frac{\pi v_F}{L} \sum_{pq} \left(u_q^2 + v_q^2 + \frac{g_q}{\pi v_F} u_q v_q \right) \rho_{pq} \rho_{p-q} + \frac{\pi (v_F - \tilde{v}_F)}{L} (N_L^2 + N_R^2) + \frac{g_0}{L} N_L N_R, \quad (25)$$

$$\tilde{v}_F = v_F \sqrt{1 - \left(\frac{g_0}{2\pi v_F} \right)^2}. \quad (26)$$

From now on we will ignore the zero mode contribution to the Hamiltonian: the above formulas clearly show that it is small ($\sim 1/L$) for a macroscopic system. Ultimately, we have:

$$UHU^\dagger = \frac{\pi v_F}{L} \sum_{pq} \left(u_q^2 + v_q^2 + \frac{g_q}{\pi v_F} u_q v_q \right) \rho_{pq} \rho_{p-q}. \quad (27)$$

This expression can be re-written in terms of fields ψ . Let us show how this is done in a technically trivial case of $\Lambda \rightarrow \infty$ (which is equivalent to $\hat{g}(x) = g_0 \delta(x)$). When Λ is infinite the functions u_q , v_q and g_q are all constants independent of q . Their values are given by (23) with $g_q \equiv g_0$. Therefore, Hamiltonian (27) is equal to

$$UHU^\dagger = \frac{\pi \tilde{v}_F}{L} \sum_{pq} \rho_{pq} \rho_{p-q}, \quad (28)$$

where \tilde{v}_F is given by (26). Finally, inverting (19), we obtain

$$UHU^\dagger = i\tilde{v}_F \int dx \sum_p p : \psi_p^\dagger \nabla \psi_p : + \text{const.} \quad (29)$$

These transformations, apart from minor differences, are equivalent to the refermionization as it is done in [4].

In certain situations, however, it is convenient to have finite Λ . Then a subtler reasoning is required. It is possible to prove that:

$$UHU^\dagger = i\tilde{v}_F \int dx \sum_p p : \psi_p^\dagger \nabla \psi_p : + (\text{irrelevant operators}). \quad (30)$$

Such proof is rather technical. It can be found in Appendix B.

Let us agree that operators with the tilde are transformations of corresponding operators without the tilde, that is $\tilde{O} = U^\dagger O U$. Then the Hamiltonian can be expressed in terms of operators $\tilde{\psi}$ as follows:

$$H = i\tilde{v}_F \int dx \sum_p p : \tilde{\psi}_p^\dagger \nabla \tilde{\psi}_p : + (\text{irrelevant operators}). \quad (31)$$

The above equation is the desired mapping of TL model on the model of quasiparticles $\tilde{\psi}$ whose interactions are small irrelevant operators. Due to the irrelevance of the interaction the low-energy properties of $\tilde{\psi}$ are those of free fermions.

Using the developed framework it is possible to discuss the effect of non-linear dispersion on the spectrum and correlations of (1). We add an extra term to the TL Hamiltonian:

$$H' = H + H_{\text{nl}}, \quad (32)$$

$$H_{\text{nl}} = \sum_p \int dx dx' \hat{h}(x - x') : \nabla \psi_p^\dagger(x) \nabla \psi_p(x') : \quad (33)$$

The subscript 'nl' stands for 'non-linear' dispersion. Function h_q which is the Fourier transform of $\hat{h}(x)$ has these properties:

$$h_q = v_F' < v_F/\Lambda \text{ for } |q| < \Lambda, \quad (34)$$

$$h_q < v_F/|q| \text{ for } |q| > \Lambda. \quad (35)$$

The second of these two conditions guarantees that for $v_F' < v_F/\Lambda$ the modified kinetic energy $p v_F q + h_q q^2$ has the same sign as the original kinetic energy $p v_F q$. That is, H_{nl} does not induce an instability of the ground state by creating spurious Fermi point.

Condition (34) implies that for small $|q| < \Lambda$ it is permissible to use

$$H_{\text{nl}} = v_F' \sum_p \int dx : \nabla \psi_p^\dagger(x) \nabla \psi_p(x) : \quad (36)$$

instead of (33). The quantity $2v'_F$ is the Fermi velocity derivative with respect to the momentum.

We need to express H_{nl} in terms of $\tilde{\psi}$. To do so it is convenient to rewrite H_{nl} with the help of density operators ρ_p rather than field operators ψ_p . This trick was already used in (19). It is easy to establish that

$$v'_F \sum_p :(\nabla\psi_p^\dagger(x))(\nabla\psi_p(x)) : - \frac{1}{6}\nabla^2\rho_p(x) = \frac{2\pi v'_F}{3} \sum_p \lim_{y \rightarrow x} \left\{ ip\rho_p(x) [:\psi_p^\dagger(y)(\nabla\psi_p(y)) : - :(\nabla\psi_p^\dagger(y))\psi_p(y) :] - 4\pi b_p(x-y)\rho_p(y) \right\}. \quad (37)$$

According to (19) the expression in the square brackets proportional to the product of two density operators. Thus:

$$\mathcal{H}_{\text{nl}}(x) = v'_F \sum_p :(\nabla\psi_p^\dagger(x))(\nabla\psi_p(x)) : - \frac{1}{6}\nabla^2\rho_p(x) = \frac{4\pi^2 v'_F}{3} \sum_p \lim_{y \rightarrow x} \lim_{z \rightarrow y} \left\{ \rho_p(x) [\rho_p(z)\rho_p(y) - b_p(z-y)] - 2b_p(x-y)\rho_p(y) \right\}. \quad (38)$$

Now we have to substitute $\rho_{pq} = u\tilde{\rho}_{pq} + v\tilde{\rho}_{-pq}$ for $q \neq 0$ in the above expression. The resultant third order polynomial of $\tilde{\rho}$'s must be rewritten in terms of $\tilde{\psi}$ with the help of (14) and (38). The final expression for H_{nl} is:

$$H_{\text{nl}} = \sum_p \int dx \left\{ \tilde{v}'_F :(\nabla\tilde{\psi}_p^\dagger)(\nabla\tilde{\psi}_p) : + ip\tilde{g}'\tilde{\rho}_{-p} \left(:\tilde{\psi}_p^\dagger(\nabla\tilde{\psi}_p) : - :(\nabla\tilde{\psi}_p^\dagger)\tilde{\psi}_p : \right) + \tilde{\mu}\tilde{\rho}_p \right\}. \quad (39)$$

The details of the derivation together with the exact formulas for coefficients \tilde{v}'_F , \tilde{g}' and $\tilde{\mu}$ can be found in Appendix C. Here we quote only the expressions which are valid if the interaction parameter g_0 is small. Let us define α_0 as $\alpha_0 = \alpha_{q=0}$. When interaction is small ($\alpha_0 \approx g_0/4\pi v_F \ll 1$) the coefficients are:

$$\tilde{v}'_F - v'_F \approx v'_F \alpha_0^2, \quad (40)$$

$$\tilde{g}' \approx 2\pi v'_F \alpha_0 = \frac{g_0 v'_F}{2v_F}, \quad (41)$$

$$\tilde{\mu} \approx \gamma \Lambda^2 v'_F \alpha_0^2, \quad (42)$$

where γ is a non-universal constant of order unity.

The total Hamiltonian H' is the sum of H , equation (31), and H_{nl} . Due to H_{nl} the quasiparticle dispersion becomes non-linear (the first term of (39)). Also, H_{nl} introduces additional interactions between the quasiparticles. The operators corresponding to these interaction (the second term in (39)) are $\mathcal{O}(v'_F g_0)$ and irrelevant. Therefore, they can be neglected provided that

$$\tilde{g}'\Lambda/v_F \ll 1 \Leftrightarrow v'_F g_0 \Lambda \ll v_F^2. \quad (43)$$

The quasiparticles remain free even if the bare dispersion is not linear as long as the bare interaction is sufficiently small. Note, that the free boson representation for TL model is much less tolerant of H_{nl} . This term is equivalent to cubic interaction between bosons (see Eq. (38) and [1]). For the bosons to remain free a stricter condition $v'_F \Lambda \ll v_F$ has to be satisfied.

Equations (31) and (39) allow us to describe the thermodynamics of the electronic liquid with the generic dispersion. For example, the free energy equals to

$$F = T \sum_{pk} \log \left(1 + e^{-\tilde{\varepsilon}_p(k)/T} \right) = \frac{LT}{\pi} \int \frac{d\tilde{\varepsilon}}{\sqrt{\tilde{v}_F^2 + 4\tilde{v}'_F \tilde{\varepsilon}}} \log \left(1 + e^{-\tilde{\varepsilon}/T} \right), \quad (44)$$

where $\tilde{\varepsilon}_p(k) = p\tilde{v}_F k + \tilde{v}'_F k^2$ is the dispersion of the quasiparticles. This result is non-perturbative in $v'_F \Lambda/v_F$. Corrections to the free energy due to the neglected quasiparticle interaction are $\mathcal{O}((v'_F g_0)^2)$. Thus, our result for F is accurate provided that (43) holds true.

The specific heat at constant chemical potential can be derived from (44) in the limit of low temperature ($T \ll \tilde{v}_F^2/\tilde{v}'_F$):

$$C(T) = \frac{\pi}{3\tilde{v}_F} T + \frac{14\pi^3 (\tilde{v}'_F)^2}{5\tilde{v}_F^5} T^3 + \mathcal{O}((\tilde{v}'_F T/\tilde{v}_F^2)^5) + \mathcal{O}(g_0^4) + \mathcal{O}((\tilde{g}')^2). \quad (45)$$

We observe that non-zero dispersion curvature generates T^3 contribution to the specific heat. (Note: irrelevant operators neglected along the way also contribute to the specific heat. However, their contribution is $\mathcal{O}(g_0^4)$ and $\mathcal{O}((\tilde{g}')^2)$. For small g_0 such corrections can be disregarded.)

The same result for the specific heat could be obtained with the help of bosonization technique. There one must expand the free energy in orders of H_{nl} . These calculations are done in Appendix D. They provide an important consistency check for the proposed approach.

3 Non-rigorous overview of the method

We have finished the introduction of our method and ready to apply it to Green's function calculations. At this point we would like to make a break between two rather technical parts and explain why the method works on intuitive level.

The central issue of this discussion could be loosely formulated as follows: why the strongly interacting bosons could be mapped on the weakly interacting fermions, what kind of 'cancellation' of interactions takes place? To answer this question we direct our attention to most essential aspects of the approach. All technical complications will be disregarded: we will ignore zero modes, normal ordering and assume that $\Lambda = \infty$. This makes the following presentation more transparent.

Let us start with the bosonized form of the TL Hamiltonian, familiar to many researchers [1]:

$$H' = H_{\text{kin}} + H_{\text{int}} + H_{\text{nl}}, \quad (46)$$

$$H_{\text{kin}} + H_{\text{int}} = \int dx \frac{\tilde{v}_F}{2} \left\{ \mathcal{K} (\nabla\Theta)^2 + \mathcal{K}^{-1} (\nabla\Phi)^2 \right\}, \quad (47)$$

$$H_{\text{nl}} = \int dx \frac{2\pi^2 v'_F}{3\sqrt{2}} \left\{ (\nabla\Theta + \nabla\Phi)^3 + (\nabla\Theta - \nabla\Phi)^3 \right\}, \quad (48)$$

where boson fields Θ and Φ are connected to the density operators in the usual way:

$$\rho_p = \frac{1}{\sqrt{2}} (\nabla\Phi + p\nabla\Theta). \quad (49)$$

Tomonaga-Luttinger liquid parameter \mathcal{K} is defined below. For the purpose of this section it is enough to remember that for small g_0 parameter \mathcal{K} is close to unity:

$$1 - \mathcal{K} = \mathcal{O}(g_0). \quad (50)$$

As we see from (48), the boson interaction is proportional to v'_F . Now we perform the Bogoliubov transformation:

$$\tilde{\Theta} = \mathcal{K}^{1/2}\Theta, \quad (51)$$

$$\tilde{\Phi} = \mathcal{K}^{-1/2}\Phi. \quad (52)$$

Our Hamiltonian becomes:

$$\begin{aligned} H' = & \int dx \frac{\tilde{v}_F}{2} \left\{ (\nabla\tilde{\Theta})^2 + (\nabla\tilde{\Phi})^2 \right\} \\ & + \frac{2\pi^2 v'_F}{3\sqrt{2}} \left\{ (\mathcal{K}^{-1/2}\nabla\tilde{\Theta} + \mathcal{K}^{1/2}\nabla\tilde{\Phi})^3 \right. \\ & \left. + (\mathcal{K}^{-1/2}\nabla\tilde{\Theta} - \mathcal{K}^{1/2}\nabla\tilde{\Phi})^3 \right\}. \end{aligned} \quad (53)$$

We rewrite it as follows:

$$H' = H_{\text{free}} + \Delta H, \quad (54)$$

$$\begin{aligned} H_{\text{free}} = & \int dx \frac{\tilde{v}_F}{2} \left\{ (\nabla\tilde{\Theta})^2 + (\nabla\tilde{\Phi})^2 \right\} \\ & + \frac{2\pi^2 v'_F}{3\sqrt{2}} \left\{ (\nabla\tilde{\Theta} + \nabla\tilde{\Phi})^3 + (\nabla\tilde{\Theta} - \nabla\tilde{\Phi})^3 \right\}, \end{aligned} \quad (55)$$

$$\begin{aligned} \Delta H = & \int dx \frac{2\pi^2 v'_F}{3\sqrt{2}} \left\{ (\mathcal{K}^{-1/2}\nabla\tilde{\Theta} + \mathcal{K}^{1/2}\nabla\tilde{\Phi})^3 \right. \\ & + (\mathcal{K}^{-1/2}\nabla\tilde{\Theta} - \mathcal{K}^{1/2}\nabla\tilde{\Phi})^3 \\ & \left. - (\nabla\tilde{\Theta} + \nabla\tilde{\Phi})^3 - (\nabla\tilde{\Theta} - \nabla\tilde{\Phi})^3 \right\}. \end{aligned} \quad (56)$$

We split the above Hamiltonian into two part for a reason. The first part H_{free} becomes a free quasiparticle Hamiltonian upon refermionization. Indeed, the first term in the curly brackets of (55) refermionizes to become the kinetic energy of the fermions with linear dispersion, the second term of (55) becomes q^2 -correction to the linear dispersion.

Refermionized Hamiltonian ΔH contains all the quasiparticle interaction. It also contains terms quadratic in $\tilde{\psi}$

which renormalize the value of v'_F . The important point, however, is that ΔH is small if g_0 is small:

$$\Delta H = \mathcal{O}((1 - \mathcal{K})v'_F) = \mathcal{O}(g_0 v'_F). \quad (57)$$

This can be established by observing that ΔH vanishes when $g_0 = 0 \Leftrightarrow \mathcal{K} = 1$. Note also, that ΔH is small if (43) is satisfied. Power counting shows that ΔH is irrelevant. Smallness and irrelevance of the quasiparticle interaction implies that the quasiparticles could be viewed as weakly interacting.

We could look at our method from a different prospective. The Hamiltonian $H' = H_{\text{kin}} + H_{\text{int}} + H_{\text{nl}}$ in addition to a quadratic (in ψ) part H_{kin} has small marginal operator H_{int} . Plus, it has an irrelevant operator H_{nl} which we want to account for non-perturbatively. Since perturbation theory in the marginal operator diverges, we must either sum certain diagrams to all orders or, as it is done in the theory of superconductivity, perform a Bogoliubov transformation which kills the undesirable marginal operator. This is what our transformation U does: $U(H_{\text{kin}} + H_{\text{int}})U^\dagger$ is quadratic in fermionic fields.

Operator H_{nl} is quadratic in ψ . In general, however, operator $UH_{\text{nl}}U^\dagger$ does not have to be quadratic. Thus, after the removal of the marginal operator by the Bogoliubov transformation, new interactions between the quasiparticles are generated. Fortunately, they are irrelevant and small. The smallness of the generated interactions becomes obvious if we observe that for small g_0 transformation U is close to the identity transformation: $U = 1 + \mathcal{O}(g_0)$. Thus, the quasiparticle interaction induced by the action of U on H_{nl} could be treated perturbatively. This is the core of the approach we proposed in the previous Section.

4 Density-density propagator

In this section we will calculate density-density Green's function. The derivation of the correlation function for the total density operator $\rho = \rho_L + \rho_R$ is quite simple if we note that ρ is proportional to the quasiparticle density operator: $\rho_q = (u_q + v_q)\tilde{\rho}_q$. Using this identity one obtains for small $|q|$:

$$\begin{aligned} \mathcal{D}_q(\tau) = & -\frac{1}{L} \langle T_\tau \{ \rho_q(\tau) \rho_{-q}(0) \} \rangle \\ = & \mathcal{K} \left(\tilde{\mathcal{D}}_{Lq}(\tau) + \tilde{\mathcal{D}}_{Rq}(\tau) \right), \end{aligned} \quad (58)$$

$$\mathcal{K} = (u_q + v_q)^2 \Big|_{q=0} = \sqrt{\frac{2\pi v_F - g_0}{2\pi v_F + g_0}}. \quad (59)$$

The chiral Green's function

$$\tilde{\mathcal{D}}_{pq}(\tau) = -\langle T_\tau \{ \tilde{\rho}_{pq}(\tau) \tilde{\rho}_{p-q}(0) \} \rangle / L$$

will be calculated below to the zeroth order in the quasiparticle interaction constant \tilde{g}' . Unlike expansion in orders of g_0 the perturbative expansion in orders of \tilde{g}' is a well-defined procedure: as we explained, the

quasiparticle-quasiparticle interaction is irrelevant. With accuracy $\mathcal{O}((\tilde{g}')^0)$ the propagator $\tilde{\mathcal{D}}_{pq}$ can be expressed as a convolution of two single-quasiparticle propagators:

$$\tilde{\mathcal{D}}_{pq}(\tau) = -\frac{1}{L} \sum_k \left\langle T_\tau \left\{ \tilde{\psi}_{p(k+q)}^\dagger(\tau) \tilde{\psi}_{p(k+q)}(0) \right\} \right\rangle \left\langle T_\tau \left\{ \tilde{\psi}_{pk}(\tau) \tilde{\psi}_{pk}^\dagger(0) \right\} \right\rangle, \quad (60)$$

$$\tilde{\mathcal{D}}_{pq\omega} = \sum_k \frac{n(\tilde{\varepsilon}_p(k)) - n(\tilde{\varepsilon}_p(k+q))}{i\omega + \tilde{\varepsilon}_p(k) - \tilde{\varepsilon}_p(k+q)}. \quad (61)$$

In equation (61) $n(\varepsilon)$ is Fermi distribution function. Thus, at $T = 0$:

$$\tilde{\mathcal{D}}_{pq\omega} = \frac{p}{4\pi\tilde{v}'_F q} \log \left\{ \frac{i\omega - p\tilde{v}_F q + \tilde{v}'_F q^2}{i\omega - p\tilde{v}_F q - \tilde{v}'_F q^2} \right\}, \quad (62)$$

$$\mathcal{D}_{q\omega} = \frac{\mathcal{K}}{4\pi\tilde{v}'_F q} \log \left\{ \frac{\omega^2 + (\tilde{v}_F q - \tilde{v}'_F q^2)^2}{\omega^2 + (\tilde{v}_F q + \tilde{v}'_F q^2)^2} \right\}. \quad (63)$$

This result has the accuracy of $\mathcal{O}((v'_F g_0)^2)$, as explained above. The omitted corrections are due to the quasiparticle interaction \tilde{g}' . At the same time, the above expression for \mathcal{D} is accurate to all orders in the dispersion curvature \tilde{v}'_F .

The retarded propagator $D_{q\omega}$ is obtained by the analytical continuation:

$$D_{q\omega} = \frac{\mathcal{K}}{4\pi\tilde{v}'_F q} \log \left\{ \frac{(\tilde{v}_F q - \tilde{v}'_F q^2)^2 - (\omega + i0)^2}{(\tilde{v}_F q + \tilde{v}'_F q^2)^2 - (\omega + i0)^2} \right\}. \quad (64)$$

For vanishing \tilde{v}'_F we can write for the Green's function the following expansion:

$$D_{q\omega} = \frac{\tilde{v}_F \mathcal{K} q^2}{\pi((\omega + i0)^2 - \tilde{v}_F^2 q^2)} + \frac{(\tilde{v}'_F)^2 \mathcal{K} q^5}{6\pi} \left(\frac{1}{(\omega - \tilde{v}_F q + i0)^3} - \frac{1}{(\omega + \tilde{v}_F q + i0)^3} \right) + \mathcal{O}((v'_F)^4) + \mathcal{O}((v'_F g_0)^2). \quad (65)$$

The first term coincides with the well-known bosonization result [2]. The second term could be also found within the bosonization approach: one has to develop a perturbation theory expansion in powers of \tilde{v}'_F . How this could be done is shown in Appendix E. These calculations serve as yet another consistency check for our method.

One can make another interesting observation when examining (65). By looking at this expansion it is impossible to guess that the propagator (64) has a branch-cut, not a pole. To determine the correct complex structure the non-perturbative in v'_F calculations are required.

Such complex structure of the propagator indicates that there is no coherently propagating bosonic mode [2]. Instead, we have a continuum of quasiparticle - quasihole pair excitations. It is possible to visualize this continuum by calculating the spectral function $B_{q\omega}$:

$$B_{q\omega} = -2\text{Im}D_{q\omega} = \frac{\mathcal{K}}{2\tilde{v}'_F q} \left\{ \vartheta(\omega^2 - (\tilde{v}_F q - \tilde{v}'_F q^2)^2) - \vartheta(\omega^2 - (\tilde{v}_F q + \tilde{v}'_F q^2)^2) \right\} \text{sgn } \omega. \quad (66)$$

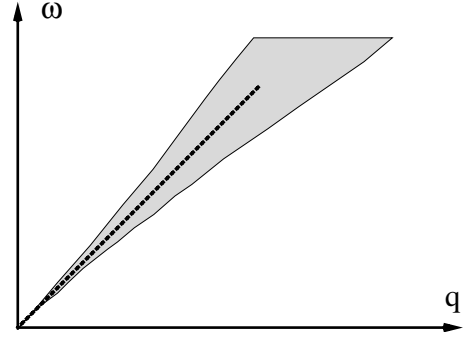


Fig. 2. When $v'_F = 0$ the spectral density of $D_{q\omega}$ is delta-function centered at $\omega = \tilde{v}_F q$ line (dash line on the figure). This is the dispersion curve of the TL bosons. For $v'_F \neq 0$ the spectral density is non-zero within the whole shaded area. This area represents the continuum of the quasiparticle-quasihole excitations. The TL bosons acquire finite life-time in such a situation.

(Note: unlike single-fermion spectral function $B_{q\omega}$ does not have to be positive.) Every point of (q, ω) plane where $B_{q\omega} \neq 0$ corresponds to a quasiparticle - quasihole excitation with total energy $|\omega|$ and total momentum q . The set of these points is shown in Figure 2.

A previous attempt to account for non-zero v'_F has been made in [1,5]. However, the bosonic representation used in the latter references is not very convenient for such a task. To illustrate the nature of the problem we consider a case of free electrons ($g_0 = 0$) with strongly non-linear dispersion $v'_F \Lambda \sim v_F$:

$$H_{\text{free}} = H_{\text{kin}} + H_{\text{nl}}. \quad (67)$$

The quasiparticle representation is trivial: $U = 1$ and $\psi = \tilde{\psi}$. The spectrum of (67) is

$$\varepsilon_k = v_F |k| + v'_F k^2. \quad (68)$$

Density-density correlation function is given by (63). In the bosonization framework operator H_{nl} corresponds to cubic interaction between bosons with the dimensionless coupling constant $v'_F \Lambda / v_F$ of order unity (see (48)). To compute either spectrum or Green's functions one must resort to the perturbation theory whose accuracy, however, is not obvious due to lack of small parameter.

If in addition to v'_F we have $g_0 \neq 0$ then the Bogoliubov rotation of the TL bosons is required. When the Bogoliubov transformation acts on H_{nl} it generates extra interaction terms of the form $\tilde{\rho}_p \tilde{\rho}_{-p}^2$. The coupling constant for this kind of interaction is of the order of $v'_F g_0$. Thus, in the bosonic representation one has to deal with two kinds of interaction terms and two coupling constants one of which is of the order of unity.

As we have seen in Section 3, if we add weak electron-electron interaction H_{int} (Eq. (3)) to H_{free} (Eq. (67)) the quasiparticle representation evolves continuously with g_0 : $U = 1 + \mathcal{O}(g_0)$, $\tilde{H}' = H_{\text{free}} + \mathcal{O}(g_0)$. Non-zero g_0 generates interaction between the quasiparticles. Yet, this interaction remains small (if (43) is fulfilled) and irrelevant.

Consequently, the Green's function is given by (63) with renormalized v_F and v'_F , the spectrum of the interacting Hamiltonian has the form (68). In short, the advantage of the quasiparticle representation stems from its ability to account for v'_F non-perturbatively.

5 Single-electron Green's function

Now we will show how to calculate the single-electron Green's function. At present we can find the Green's function for the case of $v'_F = 0$ only. Although, this Green's function has been evaluated before by a number of approaches we would like to include this derivation to demonstrate different aspects of the method.

In order to calculate the single-electron Green's function it is necessary to know how to express the electron field operator ψ^\dagger in terms of the quasiparticle operators $\tilde{\psi}^\dagger$ and $\tilde{\rho}$. The following derivation answers this question. We begin by introducing unitary operator $\exp(\varphi_p(x))$ where

$$\varphi_p(x) = \sum_{q \neq 0} \frac{\beta_{pq}}{n_q} e^{iqx} \rho_{pq}, \quad \beta_{pq} - \text{real coefficients.} \quad (69)$$

Let us calculate the commutator of this operator with ρ_{pq} :

$$\left[e^{\varphi_p(x)}, \rho_{pq} \right] = \left(e^{\varphi_p(x)} \rho_{pq} e^{-\varphi_p(x)} - \rho_{pq} \right) e^{\varphi_p(x)}. \quad (70)$$

The first term in brackets can be calculated easily:

$$e^{\varphi_p(x)} \rho_{pq} e^{-\varphi_p(x)} = \rho_{pq} + [\varphi_p(x), \rho_{pq}] = \rho_{pq} + \beta_{p-q} p e^{-iqx}. \quad (71)$$

Thus, the formula for the commutator is:

$$\left[e^{\varphi_p}, \rho_{pq} \right] = p \beta_{p-q} e^{-iqx} e^{\varphi_p}. \quad (72)$$

If we choose $\beta_{pq} = p$ the product $\psi_p^\dagger e^{\varphi_p}$ commute with $\rho_{p'q}$ for any p' and $q \neq 0$. Therefore, it commutes with Ω , equation (9). This means that such product is invariant under the action of U . The action of U on the field operator is given by:

$$\tilde{\psi}_p^\dagger = U^\dagger \psi_p^\dagger U = \psi_p^\dagger e^{\varphi_p} U^\dagger e^{-\varphi_p} U. \quad (73)$$

The latter formula is easy to invert:

$$\psi_p^\dagger(x) = \tilde{\psi}_p^\dagger(x) e^{\tilde{\varphi}_p(x)} U e^{-\tilde{\varphi}_p(x)} U^\dagger = \tilde{\psi}_p^\dagger(x) \tilde{\mathcal{F}}_p^\dagger(x), \quad (74)$$

$$\tilde{\mathcal{F}}_p^\dagger(x) = \exp \left(-p \sum_{q \neq 0} \frac{1}{n_q} e^{iqx} (w_q \tilde{\rho}_{pq} + v_q \tilde{\rho}_{-pq}) \right), \quad (75)$$

$$\left[\tilde{\psi}_p^\dagger(x), \tilde{\mathcal{F}}_p^\dagger(x) \right] = 0, \quad (76)$$

$$w_q = u_q - 1. \quad (77)$$

It gives us the desired equation for ψ^\dagger in terms of $\tilde{\psi}^\dagger$ and $\tilde{\rho}$.

Let us calculate the correlation function:

$$\langle \psi_p^\dagger(x, \tau) \psi_p(0, 0) \rangle = \left\langle \tilde{\psi}_p^\dagger(x, \tau) \tilde{\mathcal{F}}_p^\dagger(x, \tau) \tilde{\mathcal{F}}_p(0, 0) \tilde{\psi}_p(0, 0) \right\rangle. \quad (78)$$

The simplest way to evaluate this expression is to transform it in the following manner. The density operators with $pn_q < 0$ ($pn_q > 0$) must be shifted to the left (right) end, the quasiparticle field operators stay in the middle. The reason for such choice is that $\tilde{\rho}_{pq} |0\rangle = 0$ for $pn_q > 0$ and $\langle 0 | \tilde{\rho}_{pq} = 0$ for $pn_q < 0$ where $|0\rangle$ is the ground state of (1). The details of this derivation are given in Appendix F. The result is

$$\begin{aligned} \langle \psi_p^\dagger(x, \tau) \psi_p(0, 0) \rangle = \\ \left\langle \tilde{\psi}_p^\dagger(x, \tau) \tilde{\psi}_p(0, 0) \right\rangle \exp \left\{ - \sum_q (1 - e^{iqx - \tilde{v}_F |q| \tau}) \frac{v_q^2}{|n_q|} \right\} = \\ \frac{1}{2\pi(ip\tau - \tilde{v}_F \tau)} \left(\frac{a^2}{x^2 + \tilde{v}_F^2 \tau^2} \right)^\theta, \quad (79) \end{aligned}$$

where we used the notation $\theta = 2v_q^2|_{q=0} = (\mathcal{K} + 1/\mathcal{K} - 2)/2$. The same formula is derived using bosonization [1, 2].

It is well-known fact that the above Green's function does not have pole. This means that the quasiparticle state has zero overlap with the physical electron state.

6 Discussion

In this paper we solve the TL model with the help of the unitary transformation. The transformation maps the original Hamiltonian on the Hamiltonian of weakly interacting quasiparticles.

Our approach easily incorporates deviation of the electron dispersion from the linear form. As long as the bare interaction constant is sufficiently small two new non-perturbative results (Eqs. (44) and (63)) can be derived. The derived results accounts for the curvature parameter \tilde{v}'_F to all orders; they contain errors of order $(\tilde{g}')^2 \sim (v'_F g_0)^2$ due to neglected interaction among the quasiparticles.

The ability of our method to account for \tilde{v}'_F to all orders allows us to resolve the complex structure of the density-density propagator.

In principle, our diagonalization technique is not the only way to derive the quasiparticle representation. It is possible (Fig. 1) to bosonize (1) then perform the Bogoliubov transformation and then fermionize the diagonal bosonic Hamiltonian [6, 7]. Alternatively, one can construct non-fermionic excitations using appropriate exponents of bosonic operators [8]. In our case use of bosonization looks like an unnecessary detour.

To conclude, we propose a unitary transformation which maps TL model on a model of free fermions. Such approach reproduces or generalizes the TL correlation functions calculated using bosonization.

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Appendix A

In order to differentiate products of field operators one must perform normal ordering of the product under consideration. The subject of this Appendix is the definition of the normal ordering procedure we use in this paper.

Most commonly a normal ordered product of two or more field operators is defined with the reference to some non-interacting ground state $|0\rangle$. For example, for product of two operators such definition reads:

$$:\psi^\dagger(x)\psi(x'):=\psi^\dagger(x)\psi(x')-\langle 0|\psi^\dagger(x)\psi(x')|0\rangle. \quad (80)$$

The ground state is not specified in this equation. Thus, we actually have infinite number of normal ordering procedures, each corresponding to a particular state $|0\rangle$. Usually, the problem at hand dictates the choice of this state.

Normal ordered product (80) possesses two properties: (i) it is well-defined and analytical at $x = x'$; (ii) its ground state expectation value is zero.

Since the ground state of the Tomonaga-Luttinger Hamiltonian cannot be approximated by a ground state of non-interacting physical fermions ψ TL ground state cannot be used in (80). Consequently, it is necessary to explain what state we use in our definition of the normal order.

In the bosonization framework it is customary [1, 4] to normal order with respect to the the ground state of the Hamiltonian:

$$H_0 = \sum_k v_{\text{FP}} k c_k^\dagger c_k. \quad (81)$$

We accept this definition in our work as well. Thus, for a product of two field operators we have:

$$:\psi_p^\dagger(x)\psi_p(x'):=\psi_p^\dagger(x)\psi_p(x')-s_p(x-x'), \quad (82)$$

$$s_p(x) = \frac{p}{2\pi i(x-ip0)}, \quad (83)$$

where s_p is equal to $\langle 0|\psi_p^\dagger(x)\psi_p(x')|0\rangle$. In this equation the pole of $\psi_p^\dagger(x)\psi_p(x')$ at $x = x'$ is explicitly shown. The normal ordered product is well-behaved at $x = x'$ and it satisfies the property (i), formulated above.

It is not a miracle that property (i) remains intact despite the fact that the ground states of (1) and (81) are drastically different. Note that (i) is ‘‘ultraviolet’’ in its nature – it refers to short distance ($|x - x'| \ll 1/\Lambda$), high energy ($|k| \gg \Lambda$) behavior of the operator product. Since interaction (3) is limited in q -space, high-energy structure of the TL ground state is the same as that of (81). This guarantees that (i) is satisfied.

Yet, the TL ground state expectation value of (82) for generic values of x and x' is not zero. That is, property (ii) is violated. Fortunately, we never need it in this paper.

The normal ordered product of four field operators is defined by the equation:

$$\begin{aligned} \psi_p^\dagger(x)\psi_p(x')\psi_p^\dagger(y)\psi_p(y') &= :\psi_p^\dagger(x)\psi_p(x')\psi_p^\dagger(y)\psi_p(y') : \\ &+ s_p(x-y'):\psi_p(x')\psi_p^\dagger(y) : \quad (84) \\ &+ s_p(x'-y):\psi_p^\dagger(x)\psi_p(y') : \\ &+ s_p(x-y')s_p(x'-y). \quad (85) \end{aligned}$$

As above, the normal ordered product can be differentiated everywhere.

For completeness, let us define the normal ordering of the density operators $\rho_p(x)$:

$$:\rho_p(x)\rho_p(y): = \rho_p(x)\rho_p(y) - b_p(x-y), \quad (86)$$

$$\begin{aligned} :\rho_p(x)\rho_p(y)\rho_p(z): &= \rho_p(x)\rho_p(y)\rho_p(z) - b_p(x-y)\rho_p(z) \\ &- b_p(x-z)\rho_p(y) \\ &- b_p(y-z)\rho_p(x), \quad (87) \end{aligned}$$

$$b_p(x) = (s_p(x))^2. \quad (88)$$

We will use these definitions in Appendix C.

The above rules can be generalized for $T > 0$: to normal order a product of density operators at finite temperature one has to replace function $b_p(x)$ by function:

$$b_{pT}(x) = \langle \tilde{\rho}_p(x)\tilde{\rho}_p(0) \rangle = -\frac{T^2}{4\tilde{v}_F^2 \sinh^2(\pi T(x/\tilde{v}_F - ip0))}. \quad (89)$$

It is obvious that $b_{pT} = b_p$ at $T = 0$. In Appendix D we will need the finite T definition.

Appendix B

In this Appendix we provide the derivation of (30) starting from the formula (27). To express the Hamiltonian (27) in terms of ψ rather than ρ we introduce a function Δ_q :

$$\Delta_q = u_q^2 + v_q^2 + \frac{g_q}{\pi v_{\text{F}}} u_q v_q - 1 \quad (90)$$

and its Fourier transform $\hat{\Delta}(x)$:

$$\hat{\Delta}(x) = \sum_q \left(u_q^2 + v_q^2 + \frac{g_q}{\pi v_{\text{F}}} u_q v_q - 1 \right) e^{iqx}. \quad (91)$$

We defined g_q such that $g_q \rightarrow 0$ when $|q| \rightarrow \infty$. Thus, $v_q \rightarrow 0$, $u_q \rightarrow 1$ for $|q| \gg \Lambda$. Therefore, the function Δ_q vanishes for large $|q|$. This implies that $\hat{\Delta}(x)$ vanishes for $|x| \gg 1/\Lambda$. We may write:

$$\begin{aligned} UHU^\dagger &= \frac{\pi v_{\text{F}}}{L} \sum_{pq} \rho_{pq} \rho_{p-q} + \frac{\pi v_{\text{F}}}{L} \sum_{pq} \Delta_q \rho_{pq} \rho_{p-q} \\ &= i v_{\text{F}} \int dx \sum_p p : \psi_p^\dagger \nabla \psi_p : \\ &+ \frac{\pi v_{\text{F}}}{L} \sum_p \int dx dx' \hat{\Delta}(x-x') \rho_p(x) \rho_p(x'). \quad (92) \end{aligned}$$

The first term in the above equation was obtained by inversion of (19). The second term must be normal ordered first:

$$\begin{aligned} \frac{\pi v_{\text{F}}}{L} \sum_p \int dx dx' \hat{\Delta}(x-x') \rho_p(x) \rho_p(x') &= \\ \frac{\pi v_{\text{F}}}{L} \sum_p \int dx dx' \hat{\Delta}(x-x') \{ b_p(x-x') + : \rho_p(x) \rho_p(x') : \}. \quad (93) \end{aligned}$$

The additive constant proportional to $\int dx dx' \hat{\Delta} b_p$ will be disregarded. The normal ordered product of two ρ 's can be expanded into Taylor series in powers of $(x - x')$:

$$\begin{aligned} \sum_p \int dx dx' \hat{\Delta}(x - x') : \rho_p(x) \rho_p(x') : = \\ \sum_p \left[\int dx' \hat{\Delta}(x') \right] \int dx : \rho_p^2(x) : \\ + \frac{1}{2} \left[\int dx' x'^2 \hat{\Delta}(x') \right] \int dx : \rho_p(x) \nabla^2 \rho_p(x) : + \dots, \end{aligned} \quad (94)$$

where ellipsis stand for higher order terms of the Taylor series. Since $\hat{\Delta}(x)$ vanishes for large $|x|$ the integrals of $\hat{\Delta}$ are well-defined. We did not show the term proportional to $: \rho_p \nabla \rho_p :$ since it is total derivative $\nabla : \rho_p^2 :$, thus, it vanishes upon integration.

We notice that the first term of expansion (94) is marginal. All other terms are irrelevant. Indeed, power counting shows that the scaling dimension of the operator $: \rho_p \nabla^2 \rho_p :$ is $4 > 2$. The scaling dimensions of the omitted terms are even higher. Thus, it is permissible to write:

$$\begin{aligned} \frac{\pi v_F}{L} \sum_p \int dx dx' \hat{\Delta}(x - x') \rho_p(x) \rho_p(x') = \\ \pi v_F \Delta_0 \sum_p \int dx : \rho_p^2(x) : + (\text{irrelevant operators}), \end{aligned} \quad (95)$$

$$\Delta_0 = \Delta_q|_{q=0}. \quad (96)$$

Consequently, transforming $: \rho_p^2 :$ into the kinetic energy operator with the help of (19) and substituting the result into (92), we get the desired equation:

$$\begin{aligned} U H U^\dagger = \\ \int dx \sum_p i p \tilde{v}_F : \psi_p^\dagger \nabla \psi_p : + (\text{irrelevant operators}), \end{aligned} \quad (97)$$

$$\begin{aligned} \tilde{v}_F = v_F (1 + \Delta_0) = v_F \left(u_q^2 + v_q^2 + \frac{g_q}{\pi v_F} u_q v_q \right) \Big|_{q=0} \\ = v_F \sqrt{1 - \left(\frac{g_0}{2\pi v_F} \right)^2}. \end{aligned} \quad (98)$$

Note, that the omitted operators, in addition to being irrelevant, are also small ($\sim \Delta_0 \sim g_0^2/v_F^2$) if interaction is small $g_0 \ll v_F$.

Appendix C

In this Appendix we show how H_{nl} can be expressed in terms of the quasiparticle field operators $\tilde{\psi}^\dagger, \tilde{\psi}$. We start

with expression for the local density \mathcal{H}_{nl} :

$$\begin{aligned} \mathcal{H}_{\text{nl}}(x) = v'_F \sum_p : (\nabla \psi_p^\dagger(x)) (\nabla \psi_p(x)) : - \frac{1}{6} \nabla^2 \rho_p(x) \\ = \frac{4\pi^2 v'_F}{3} \sum_p \lim_{z \rightarrow x} \lim_{y \rightarrow x} \left\{ \rho_p(z) [\rho_p(x) \rho_p(y) - b_p(x - y)] \right. \\ \left. - 2b_p(z - y) \rho_p(y) \right\}. \end{aligned} \quad (99)$$

The last line can be abbreviated if one use the definition of the normal ordering for the density operators $\rho_p(x)$, equation (87). With this notation we can rewrite the formula for \mathcal{H}_{nl} :

$$\mathcal{H}_{\text{nl}} = \frac{4\pi^2 v'_F}{3} \sum_p : \rho_p^3 :. \quad (100)$$

To express the density operator $\rho_p(x)$ in terms of the quasiparticle density operators $\tilde{\rho}_p(x)$ we must remember that:

$$\rho_{pq} = u_q \tilde{\rho}_{pq} + v_q \tilde{\rho}_{-pq} \text{ for } q \neq 0, \quad (101)$$

$$N_p = \tilde{N}_p \text{ for zero modes.} \quad (102)$$

Therefore, the density operator in co-ordinate space is equal to:

$$\rho_p(x) = \delta \tilde{\rho}_{p0} + \tilde{\rho}_p^u(x) + \tilde{\rho}_{-p}^v(x), \quad (103)$$

$$\delta \tilde{\rho}_{p0} = \left((1 - u_q) \tilde{N}_p - v_q \tilde{N}_{-p} \right) \Big|_{q=0}, \quad (104)$$

where $\tilde{\rho}_p^u$ is the convolution of the density operator $\tilde{\rho}_p(x)$ with $\hat{u}(x)$:

$$\tilde{\rho}_p^u(x) = (\hat{u} * \tilde{\rho}_p)(x) = \int dx' \hat{u}(x - x') \tilde{\rho}_p(x'), \quad (105)$$

$$\hat{u}(x) = \int \frac{dq}{2\pi} u_q e^{iqx}. \quad (106)$$

Likewise, $\tilde{\rho}_{-p}^v = (\hat{v} * \tilde{\rho}_{-p})(x)$, where function $\hat{v}(x)$ is defined in the same manner as $\hat{u}(x)$. It is tempting to equate \mathcal{H}_{nl} and

$$\frac{4\pi^2 v'_F}{3} \sum_p : (\delta \tilde{\rho}_{p0} + \tilde{\rho}_p^u + \tilde{\rho}_{-p}^v)^3 :. \quad (107)$$

This, however, is not exactly accurate since the normal ordering and Bogoliubov transformation do not “commute” with each other. Such phenomena becomes obvious if we were to think in terms of Bose creation and annihilation operators. Normal ordering places all creation operators to the left of all annihilation operators. Bogoliubov transformation mixes creation and annihilation operators, thus, it spoils normal ordering.

To rewrite \mathcal{H}_{nl} in terms of $\tilde{\rho}^{u,v}$ correctly let us first examine the expression in square brackets in equation (99):

$$\begin{aligned} & \rho_p(x)\rho_p(y) - b_p(x-y) = : \rho_p(x)\rho_p(y) : = \\ & (\delta\tilde{\rho}_{p0} + \tilde{\rho}_p^u(x) + \tilde{\rho}_{-p}^v(x)) (\delta\tilde{\rho}_{p0} + \tilde{\rho}_p^u(y) + \tilde{\rho}_{-p}^v(y)) \\ & - b_p(x-y) = \\ & \left\{ : \tilde{\rho}_p^u(x)\tilde{\rho}_p^u(y) : + (\hat{u} * b_p * \hat{u})(x-y) - b_p(x-y) \right\} \\ & + \left\{ : \tilde{\rho}_{-p}^v(x)\tilde{\rho}_{-p}^v(y) : + (\hat{v} * b_{-p} * \hat{v})(x-y) \right\} \\ & + \left\{ \tilde{\rho}_p^u(x)\tilde{\rho}_{-p}^v(y) + \tilde{\rho}_{-p}^v(x)\tilde{\rho}_p^u(y) \right\} \\ & + \delta\tilde{\rho}_{p0} \left\{ \tilde{\rho}_p^u(x) + \tilde{\rho}_{-p}^v(x) + \tilde{\rho}_p^u(y) + \tilde{\rho}_{-p}^v(y) \right\} + \delta\tilde{\rho}_{p0}^2, \end{aligned} \quad (108)$$

where we applied the following transformations:

$$\begin{aligned} & \tilde{\rho}_p^u(x)\tilde{\rho}_p^u(y) = \\ & \int dx' dy' u(x-x')u(y-y')\tilde{\rho}_p(x')\tilde{\rho}_p(y') = \\ & \int dx' dy' u(x-x')u(y-y') (: \tilde{\rho}_p(x')\tilde{\rho}_p(y') : + b_p(x'-y')) = \\ & : \tilde{\rho}_p^u(x)\tilde{\rho}_p^u(y) : + (\hat{u} * b_p * \hat{u}). \end{aligned} \quad (109)$$

Similar transformations could be done for $\tilde{\rho}^v(x)\tilde{\rho}^v(y)$. Since $u_q^2 - v_q^2 = 1$ the identity:

$$\hat{u} * f * \hat{u} = f + (\hat{v} * f * \hat{v}) \quad (110)$$

holds true. Therefore:

$$\begin{aligned} & : \rho_p(x)\rho_p(y) : = : (\tilde{\rho}_p^u(x) + \tilde{\rho}_{-p}^v(x)) (\tilde{\rho}_p^u(y) + \tilde{\rho}_{-p}^v(y)) : \\ & + (\hat{v} * (b_p + b_{-p}) * \hat{v})(x-y) \\ & + \delta\tilde{\rho}_{p0} \left\{ \tilde{\rho}_p^u(x) + \tilde{\rho}_{-p}^v(x) + \tilde{\rho}_p^u(y) + \tilde{\rho}_{-p}^v(y) \right\} + \delta\tilde{\rho}_{p0}^2. \end{aligned} \quad (111)$$

The extra terms in this formula are non-singular functions of $(x-y)$. In particular:

$$(\hat{v} * (b_p + b_{-p}) * \hat{v})(x-y) = \int \frac{dq}{(2\pi)^2} e^{iq(x-y)} |q| v_q^2. \quad (112)$$

Observe that the normal ordered product of ρ 's differs from the normal ordered product of $\tilde{\rho}$'s by a non-singular operator.

Generalizing the above calculations for the product of three ρ 's we obtain the following expression for \mathcal{H}_{nl} :

$$\begin{aligned} \mathcal{H}_{\text{nl}} = & \frac{4\pi^2 v'_F}{3} \sum_p : (\tilde{\rho}_p^u)^3 : + : (\tilde{\rho}_{-p}^v)^3 : + 3\tilde{\rho}_p^u : (\tilde{\rho}_{-p}^v)^2 : \\ & + 3\tilde{\rho}_{-p}^v : (\tilde{\rho}_p^u)^2 : + \tilde{c} (\delta\tilde{\rho}_{p0} + \tilde{\rho}_p^u + \tilde{\rho}_{-p}^v) + (\text{z.m.}), \end{aligned} \quad (113)$$

$$\tilde{c} = \frac{3}{4\pi^2} \int dq |q| v_q^2, \quad (114)$$

where ‘‘z.m.’’ stand for zero modes terms which vanish in the thermodynamic limit.

For small momenta the above formula can be simplified. What must be done, in its substance, amounts to inversion of equations (100) and (19). To illustrate this statement, consider the first and the second terms of (113).

Limiting ourselves to the case $\Lambda = \infty$ we can neglect the q -dependence of u_q and v_q . Then, $\tilde{\rho}_p^u = u\tilde{\rho}_p$, $\tilde{\rho}_p^v = v\tilde{\rho}_p$ and it is possible to write:

$$\begin{aligned} \frac{4\pi^2 v'_F}{3} \sum_p : (\tilde{\rho}_p^u)^3 : + : (\tilde{\rho}_p^v)^3 : = & \frac{4\pi^2 v'_F}{3} (u^3 + v^3) \sum_p : \tilde{\rho}_p^3 : = \\ & v'_F (u^3 + v^3) \left(: \nabla \tilde{\psi}_p^\dagger \nabla \tilde{\psi}_p : - \frac{1}{6} \nabla^2 \tilde{\rho}_p \right). \end{aligned} \quad (115)$$

When handling the third and the fourth terms of (113) we are to act in the similar fashion: the product $: \tilde{\rho}_p^2 :$ should be rewritten in terms of $\tilde{\psi}$ with the help of (19). These two terms give the quasiparticle interaction.

For a finite value of Λ our task becomes somewhat more complicated. In such a situation one can generalize the procedure of Appendix B. Let us briefly discuss the core of this generalization. As an example, consider the expression:

$$\begin{aligned} : (\tilde{\rho}_p^v)^3 : = & \int dx' dx'' dx''' v(x-x')v(x-x'')v(x-x''') \\ & \times : \tilde{\rho}_p(x')\tilde{\rho}_p(x'')\tilde{\rho}_p(x''') : . \end{aligned} \quad (116)$$

It is easy to show with the help of Taylor expansion (see Appendix B) that the normal ordered product $: \tilde{\rho}_p(x')\tilde{\rho}_p(x'')\tilde{\rho}_p(x''') :$ is equal to

$$: \tilde{\rho}_p^3(x) : + (\text{more irrelevant operators}),$$

where ‘‘more irrelevant operators’’ stands for operators whose scaling dimension is bigger than 3. Therefore, we obtain the following equation:

$$: (\tilde{\rho}_p^v)^3 : = a : \tilde{\rho}_p^3 : + (\text{more irrelevant operators}), \quad (117)$$

where coefficient a is equal to $v_q^3|_{q=0}$. The expression $: \tilde{\rho}_p^3 :$ has to be transformed further as in equation (115). The remaining terms of (113) are transformed similarly. Therefore, one establishes:

$$\begin{aligned} \mathcal{H}_{\text{nl}} = & \sum_p \left\{ \tilde{v}'_F : (\nabla \tilde{\psi}_p^\dagger) (\nabla \tilde{\psi}_p) : - \frac{\tilde{v}'_F}{6} \nabla^2 \tilde{\rho}_p \right. \\ & \left. + ip\tilde{g}' \tilde{\rho}_{-p} \left(: \tilde{\psi}_p^\dagger (\nabla \tilde{\psi}_p) : - : (\nabla \tilde{\psi}_p^\dagger) \tilde{\psi}_p : \right) \right\} \end{aligned} \quad (118)$$

$$+ \tilde{\mu} (\tilde{\rho}_L + \tilde{\rho}_R) + (\text{more irrelevant operators}), \quad (119)$$

$$\tilde{v}'_F = v'_F (u_q^3 + v_q^3)|_{q=0}, \quad (120)$$

$$\tilde{g}' = 2\pi v'_F (u_q^2 v_q + u_q v_q^2)|_{q=0}, \quad (121)$$

$$\tilde{\mu} = v'_F \left(\int dq |q| v_q^2 \right). \quad (122)$$

The values of u_q and v_q are given by (23). Finally, we find for the Hamiltonian:

$$\begin{aligned} H_{\text{nl}} = & \sum_p \int dx \left\{ \tilde{v}'_F : (\nabla \tilde{\psi}_p^\dagger) (\nabla \tilde{\psi}_p) : \right. \\ & \left. + ip\tilde{g}' \tilde{\rho}_{-p} \left(: \tilde{\psi}_p^\dagger (\nabla \tilde{\psi}_p) : - : (\nabla \tilde{\psi}_p^\dagger) \tilde{\psi}_p : \right) \right. \\ & \left. + \tilde{\mu} \tilde{\rho}_p \right\} + (\text{more irrelevant operators}). \end{aligned} \quad (122)$$

We omitted the full derivative $\nabla^2 \rho$ from the Hamiltonian since it contributes on the system boundary only.

We don't want to prove here that the quasiparticle interaction produces no ultraviolet divergence of the perturbation theory. Instead, this issue is examined at the end of Appendix F. This is because the formalism developed there is particularly convenient for discussion of ultraviolet properties of the quasiparticle interaction.

Finally, let us briefly explain, why the quasiparticle interaction generated by the action of U on H_{nl} is small. This explanation is equivalent to the one given in Section 3. Initially, our H_{nl} could be expressed as a cube of $\rho_p(x)$, equation (100). (Although, superficially, this expression appears to be sixth order in ψ_p , a subtler analysis shows that it is quadratic in ψ_p , Eq. (99).) The Bogoliubov transformation U converts (100) into (113). Unlike (100), equation (113) is indeed a sixth order polynomial in ψ_p . However, for small g_0 operator U is close to the identity transformation: $U = 1 + \mathcal{O}(g_0)$. Therefore,

$$v'_F : \rho_p^3 : = v'_F : \tilde{\rho}_p^3 : + \mathcal{O}(g_0 v'_F). \quad (123)$$

The first term on the right is proportional to $\nabla \tilde{\psi}_p^\dagger \nabla \tilde{\psi}_p$, equation (100). All generated terms, such as quasiparticle interactions and corrections to v'_F , are $\mathcal{O}(g_0 v'_F)$.

Appendix D

In this Appendix we show how the result for the specific heat, equation (45), can be obtained within the bosonization framework. More specifically, $\mathcal{O}((v'_F)^2)$ correction to the specific heat will be calculated with the help of the perturbation theory for the free boson Hamiltonian:

$$H = \frac{\pi \tilde{v}_F}{2} \sum_{pq} \tilde{\rho}_{pq} \tilde{\rho}_{p-q}. \quad (124)$$

We will see that it is the same as $\mathcal{O}((v'_F)^2)$ term in equation (45).

Our first step is to rewrite the non-linear dispersion Hamiltonian H_{nl} in a form particularly suitable for $T > 0$ calculations: since in this Appendix we work at finite temperature it is natural to express H_{nl} with the help of the finite temperature generalization of (86) and (87). As it was explained, such generalization is achieved by placing b_{pT} , equation (89), instead of b_p , equation (88).

Using the new rules of the normal ordering we could prove that at $T \geq 0$ the following is true:

$$\begin{aligned} \tilde{v}'_F \sum_p : (\nabla \tilde{\psi}_p^\dagger(x)) (\nabla \tilde{\psi}_p(x)) : - \frac{1}{6} \nabla^2 \tilde{\rho}_p(x) = \\ \frac{4\pi^2 \tilde{v}'_F}{3} \sum_p : \tilde{\rho}_p^3 : + \frac{T^2}{4\tilde{v}_F^2} \tilde{\rho}_p, \end{aligned} \quad (125)$$

where $: \dots :$ denotes here the finite temperature normal order. The proof of this equation is simple. Start from

equation (38):

$$\begin{aligned} \tilde{v}'_F \sum_p : (\nabla \tilde{\psi}_p^\dagger(x)) (\nabla \tilde{\psi}_p(x)) : - \frac{1}{6} \nabla^2 \tilde{\rho}_p(x) = \quad (126) \\ \frac{4\pi^2 \tilde{v}'_F}{3} \sum_p \lim_{y \rightarrow x} \lim_{z \rightarrow y} \left\{ \tilde{\rho}_p(x) [\tilde{\rho}_p(z) \tilde{\rho}_p(y) - b_p(z-y)] \right. \\ \left. - 2b_p(x-y) \tilde{\rho}_p(y) \right\} = \\ \frac{4\pi^2 \tilde{v}'_F}{3} \sum_p \lim_{y \rightarrow x} \lim_{z \rightarrow y} \left\{ \tilde{\rho}_p(x) [\tilde{\rho}_p(z) \tilde{\rho}_p(y) - b_{pT}(z-y)] \right. \\ \left. - 2b_{pT}(x-y) \tilde{\rho}_p(y) \right\} \\ + 3\tilde{\rho}_p(x) \lim_{y \rightarrow x} \{ b_{pT}(x-y) - b_p(x-y) \}. \end{aligned}$$

The last limit is

$$\lim_{y \rightarrow x} \{ b_{pT}(x-y) - b_p(x-y) \} = \frac{T^2}{12\tilde{v}_F^2}. \quad (127)$$

Thus, bosonized H_{nl} equals to

$$H_{\text{nl}} = \frac{4\pi^2 \tilde{v}'_F}{3} \sum_p \int dx \left(: \tilde{\rho}_p^3 : + \frac{T^2}{4\tilde{v}_F^2} \tilde{\rho}_p \right) + \mathcal{O}(g_0 v'_F). \quad (128)$$

This formula does not imply that H_{nl} depends on the temperature. In this Appendix the symbol $: \dots :$ denotes the finite T generalization of the normal ordering, consequently, the expression $: \tilde{\rho}_p^3 :$ varies with temperature. The T^2 term compensates the temperature dependence of the normal ordered product $: \tilde{\rho}_p^3 :$ so that H_{nl} is temperature independent as it must be. We remind the reader that such a way of writing is a matter of convenience.

In the above expression we ignore the quasiparticle interaction and the correction to the chemical potential since we are interested in $\mathcal{O}((v'_F)^2)$ corrections only while the neglected terms contribute to the thermodynamics at $\mathcal{O}((v'_F g_0)^2)$.

The correction to the free energy due to H_{nl} is given by the usual perturbation theory formula:

$$\begin{aligned} \delta F = -\frac{1}{2} \int d\tau \langle H_{\text{nl}}(\tau) H_{\text{nl}}(0) \rangle = \\ - (\tilde{v}'_F)^2 L \int d\tau dx \left(\frac{16\pi^4}{9} \langle : \tilde{\rho}_L^3(x, \tau) : : \tilde{\rho}_L^3(0, 0) : \rangle \right. \\ \left. + \frac{\pi^4 T^4}{9\tilde{v}_F^4} \langle \tilde{\rho}_L(x, \tau) \tilde{\rho}_L(0, 0) \rangle \right). \end{aligned} \quad (129)$$

Using Wick's theorem we find:

$$\langle : \tilde{\rho}_L^3(x, \tau) : : \tilde{\rho}_L^3(0, 0) : \rangle = 6 \langle \tilde{\rho}_L(x, \tau) \tilde{\rho}_L(0, 0) \rangle^3, \quad (130)$$

$$\langle \tilde{\rho}_L(x, \tau) \tilde{\rho}_L(0, 0) \rangle = -\frac{T^2}{4\tilde{v}_F^2 \sinh^2(\pi T(x/\tilde{v}_F - i\tau))}. \quad (131)$$

Therefore:

$$\begin{aligned} \delta F = (\tilde{v}'_F)^2 L \int d\tau dx \left(\frac{\pi^4 T^6}{6\tilde{v}_F^6} \frac{1}{\sinh^6(\pi T(x/\tilde{v}_F - i\tau))} \right. \\ \left. + \frac{\pi^4 T^6}{36\tilde{v}_F^6} \frac{1}{\sinh^2(\pi T(x/\tilde{v}_F - i\tau))} \right). \end{aligned} \quad (132)$$

It is easy to calculate the integral

$$\int_{-\infty}^{+\infty} \frac{dx}{\sinh^2(\pi T(x/\tilde{v}_F - i\tau))} = -\frac{2\tilde{v}_F}{\pi T} \quad (133)$$

for $0 < \tau < \beta$. To find the integral of $1/\sinh^6$ we use a trick. One can check directly that:

$$\frac{1}{\sinh^6 r} = \frac{8}{15} \frac{1}{\sinh^2 r} + \frac{d^2 I(r)}{dr^2}, \quad (134)$$

$$I(x) = \frac{1}{20} \frac{1}{\sinh^4 r} - \frac{2}{15} \frac{1}{\sinh^2 r}. \quad (135)$$

The integral of the full derivative is zero and we establish for $0 < \tau < \beta$

$$\int_{-\infty}^{+\infty} \frac{dx}{\sinh^6(\pi T(x/\tilde{v}_F - i\tau))} = \frac{8}{15} \int_{-\infty}^{+\infty} \frac{dx}{\sinh^2(\pi T(x/\tilde{v}_F - i\tau))} = -\frac{16\tilde{v}_F}{15\pi T}. \quad (136)$$

Therefore, one finds the following corrections to the free energy, the entropy and the specific heat:

$$\delta F/L = -\frac{7\pi^3(\tilde{v}'_F)^2 T^4}{30\tilde{v}_F^5}, \quad (137)$$

$$\delta S/L = -\frac{\partial(\delta F/L)}{\partial T} = \frac{14\pi^3(\tilde{v}'_F)^2 T^3}{15\tilde{v}_F^5}, \quad (138)$$

$$\delta C = T \frac{\partial(\delta S/L)}{\partial T} = \frac{14\pi^3(\tilde{v}'_F)^2 T^3}{5\tilde{v}_F^5}. \quad (139)$$

The last expression coincides with the second term of equation (45) as it should be.

Appendix E

In this Appendix we calculate $\mathcal{O}((v'_F)^2)$ correction to the zero-temperature boson propagator. Such correction must coincide with the second term of (65).

The correction in question to the chiral Matsubara propagator comes from H_{nl} , equation (128), with $T = 0$. The appropriate Feynman diagram is shown in Figure 3. As one can see from equation (58) the corresponding expression is:

$$\begin{aligned} \delta \mathcal{D}_p(x, \tau) &= -\frac{\mathcal{K}}{2} \left(\frac{4\pi^2 \tilde{v}'_F}{3} \right)^2 \int d\tau' d\tau'' dx' dx'' \langle T_\tau \{ \tilde{\rho}_p(0, 0) \\ &\times : \tilde{\rho}_p^3(x', \tau') : : \tilde{\rho}_p^3(x'', \tau'') : \tilde{\rho}_p(x, \tau) \} \rangle_{\text{connected}} + \mathcal{O}((v'_F g_0)^2). \end{aligned} \quad (140)$$

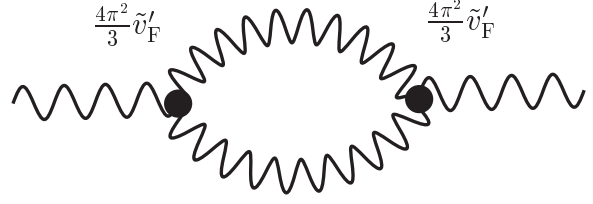


Fig. 3. Correction to $\tilde{\mathcal{D}}_{pq}^0$ due to the dispersion curvature. Wavy lines correspond to $\tilde{\mathcal{D}}^0$, the vertexes are proportional to \tilde{v}'_F [9].

The time-ordered average with respect to Hamiltonian (124) is equal to:

$$\begin{aligned} &\int d\tau' d\tau'' dx' dx'' \langle T_\tau \{ \tilde{\rho}_p(0, 0) \\ &\times : \tilde{\rho}_p^3(x', \tau') : : \tilde{\rho}_p^3(x'', \tau'') : \tilde{\rho}_p(x, \tau) \} \rangle_{\text{connected}} = \\ &3 \times 3 \times 2 \times 2 \int d\tau' d\tau'' dx' dx'' \tilde{\mathcal{D}}_p^0(\tau', x') \\ &\times \left(\tilde{\mathcal{D}}_p^0(\tau'' - \tau', x'' - x') \right)^2 \tilde{\mathcal{D}}_p^0(\tau - \tau'', x - x''), \end{aligned} \quad (141)$$

where $\tilde{\mathcal{D}}_p^0(x, \tau)$ is the chiral free boson Green's function $\tilde{\mathcal{D}}_p^0(x, \tau) = -(1/L) \langle T_\tau \{ \tilde{\rho}_p(0, 0) \tilde{\rho}_p(x, \tau) \} \rangle$ for Hamiltonian (124).

The numerical factor on the right-hand side of (141) corresponds to 36 possible ways of contracting the density operators into a connected diagram presented in Figure 3. First, the external operator $\tilde{\rho}_p(0, 0)$ could contract with either of two vertexes; the operator $\tilde{\rho}_p(x, \tau)$ contracts with the remaining vertex. This gives a factor of two. Second, a given external operator could contract with either of three density operators in a vertex. This gives a factor of three for one external operator and another factor of three for the second external operator. Finally, after contracting with the external operators, every vertex has two uncontracted density operators. For them there are two ways of contraction. Thus, we have yet another factor of two.

In the Fourier space we can write:

$$\mathcal{D}_{pq\omega} \approx \mathcal{K} \tilde{\mathcal{D}}_{pq\omega}^0 + 32\pi^4 \mathcal{K} (\tilde{v}'_F)^2 \left(\tilde{\mathcal{D}}_{pq\omega}^0 \right)^2 \tilde{\Pi}_{pq\omega}, \quad (142)$$

$$\tilde{\mathcal{D}}_{pq\omega}^0 = \frac{1}{2\pi} \frac{pq}{i\omega - p\tilde{v}_F q}, \quad (143)$$

$$\begin{aligned} \tilde{\Pi}_{pq\omega} &= -T \sum_{\Omega} \int \frac{dk}{2\pi} \tilde{\mathcal{D}}_{pk\Omega}^0 \tilde{\mathcal{D}}_{p(q-k)(\omega-\Omega)}^0 \\ &= \frac{1}{48\pi^3} \frac{pq^3}{i\omega - p\tilde{v}_F q}. \end{aligned} \quad (144)$$

Therefore, for correction to the full propagator we write:

$$\begin{aligned} \delta \mathcal{D}_{q\omega} &= \sum_p \delta \mathcal{D}_{pq\omega} \\ &= \frac{\mathcal{K} (\tilde{v}'_F)^2 q^5}{6\pi} \left(\frac{1}{(i\omega - \tilde{v}_F q)^3} - \frac{1}{(i\omega + \tilde{v}_F q)^3} \right). \end{aligned} \quad (145)$$

After analytical continuation we recover the $\mathcal{O}((v'_F)^2)$ term of (65).

Appendix F

In this Appendix we provide a detailed derivation of the single-electron Green's function for TL model, equation (79). The formalism developed for the Green's function is also a convenient tool to prove that the quasiparticle perturbation theory has no ultraviolet divergences.

We start with equation (78). First, the product $\tilde{\mathcal{F}}_p^\dagger \tilde{\mathcal{F}}_p$ has to be normal-ordered:

$$\tilde{\mathcal{F}}_p^\dagger(x, \tau) \tilde{\mathcal{F}}_p(0, 0) = e^{\mathbf{W}_p(x+ipy, 0)} e^{-\mathbf{W}_p^\dagger(x+ipy, 0)} \times e^{\mathbf{V}_p(x-ipy, 0)} e^{-\mathbf{V}_p^\dagger(x-ipy, 0)} \quad (146)$$

$$\times e^{-p \sum_q \frac{1}{n_q} (1 - e^{iqx - pqy}) w_q^2 \vartheta(pq)} e^{p \sum_q \frac{1}{n_q} (1 - e^{iqx + pqy}) v_q^2 \vartheta(-pq)},$$

$$\mathbf{W}_p(x, x') = p \sum_q \frac{1}{n_q} \left(e^{iqx'} - e^{iqx} \right) w_q \tilde{\rho}_{pq} \vartheta(-pq), \quad (147)$$

$$\mathbf{V}_p(x, x') = p \sum_q \frac{1}{n_q} \left(e^{iqx'} - e^{iqx} \right) v_q \tilde{\rho}_{-pq} \vartheta(pq). \quad (148)$$

Here $y = \tilde{v}_F \tau$. This equation was derived with the help of commutation relations (5). In order to calculate $\tilde{\rho}(\tau)$ we used the expression (20).

Second, we transform the whole expression $\tilde{\psi}_p^\dagger(x, \tau) \tilde{\mathcal{F}}_p^\dagger(x, \tau) \tilde{\mathcal{F}}_p(0, 0) \tilde{\psi}_p(0, 0)$. We shift $\exp(\mathbf{W}_p)$ to the left past the quasiparticle field operator $\tilde{\psi}_p^\dagger$. The exponent $\exp(-\mathbf{W}_p^\dagger)$ is shifted to the very right. To perform these shifts the commutation rule (6) has to be used.

$$\tilde{\psi}_p^\dagger(x, \tau) \tilde{\mathcal{F}}_p^\dagger(x, \tau) \tilde{\mathcal{F}}_p(0, 0) \tilde{\psi}_p(0, 0) = \left\{ e^{\mathbf{W}_p(x+ipy, 0)} \left(\tilde{\psi}_p^\dagger(x, \tau) \tilde{\psi}_p(0, 0) \right) e^{-\mathbf{W}_p^\dagger(x+ipy, 0)} \right\} \times \left(e^{\mathbf{V}_p(x-ipy, 0)} e^{-\mathbf{V}_p^\dagger(x-ipy, 0)} \right) \times e^{-p \sum_q \frac{1}{n_q} (1 - e^{iqx - pqy}) (w_q^2 + 2w_q) \vartheta(pq)} \times e^{p \sum_q \frac{1}{n_q} (1 - e^{iqx + pqy}) v_q^2 \vartheta(-pq)}.$$

After taking the expectation value of this expression and recalling that $w_q^2 + 2w_q = v_q^2$ we obtain the desired expression for the correlation function.

The above formula sets a convenient background for a rather general discussion of the ultraviolet properties of the quasiparticle-quasiparticle interactions. Imagine that we add the following term to the TL Hamiltonian:

$$H_{\text{extra}} = \sum_p \int dx dx' \hat{Z}(x - x') \psi_p^\dagger(x) \psi_p(x'). \quad (150)$$

Operator H_{nl} is a particular case of H_{extra} with $\hat{Z} = -\hat{h}''$.

With the help of (149) it is easy to rewrite H_{extra} in terms of the quasiparticle operators $\tilde{\psi}$ and $\tilde{\rho}$:

$$H_{\text{extra}} = \sum_p \int dx dx' \hat{Z}(x - x') \zeta(x - x') \times \left\{ e^{\mathbf{W}_p(x, x')} \left(\tilde{\psi}_p^\dagger(x) \tilde{\psi}_p(x') \right) e^{-\mathbf{W}_p^\dagger(x, x')} \right\} \times \left(e^{\mathbf{V}_{-p}(x, x')} e^{-\mathbf{V}_{-p}^\dagger(x, x')} \right), \quad (151)$$

$$\zeta(x) = e^{\sum_q \frac{1}{|n_q|} (1 - e^{iqx}) v_q^2}. \quad (152)$$

Function $\zeta(x)$ is well-defined for $|x| \ll 1/\Lambda$ and vanishes algebraically for large $|x|$. Operator (151) is normal-ordered, therefore, it is safe to expand the exponentials:

$$H_{\text{extra}} = \sum_{n, m, n', m'} H_{nmn'm'} = \sum_{n, m, n', m'} \int dx dx' \frac{\hat{Z} \zeta}{n! m! n'! m'!} \times \mathbf{W}_p^n \tilde{\psi}_p^\dagger \tilde{\psi}_p (-\mathbf{W}_p^\dagger)^m \mathbf{V}_p^{n'} (-\mathbf{V}_p^\dagger)^{m'}. \quad (153)$$

Since $\mathbf{W} = \mathcal{O}(g_0^2)$ and $\mathbf{V} = \mathcal{O}(g_0)$ such expansion is controlled by the smallness of g_0 . We want to argue that any interaction term $H_{nmn'm'}$, $n+m+n'+m' > 0$ does not lead to ultraviolet divergence of perturbation theory. Consider, for example, the second order correction to the ground state energy:

$$\delta E_{nn'} = - \sum_N \sum_{\substack{p_1 \dots p_N \\ k_1 \dots k_N}} \frac{\left| \langle \substack{p_1 \dots p_N \\ k_1 \dots k_N} | H_{n0n'0} | 0 \rangle \right|^2}{\sum_i \tilde{\epsilon}(p_i) + \tilde{\epsilon}(k_i)} \delta \left(\sum_i p_i + k_i \right). \quad (154)$$

Summation in the above formula goes over $N \leq n + n' + 1$ quasiparticle momenta p_i and over N quasihole momenta k_i . We include $m = m' = 0$ terms only because other terms annihilate the ground state. Thus, $m, m' \neq 0$ terms do not contribute to the second order ground state correction. They do, however, contribute to higher order corrections and to corrections to different propagators.

If the matrix element does not vanish at large momenta the expression (154) suffers from the ultraviolet divergence. We will show that the matrix element goes to zero if at least one $|p_i|$ or $|k_i|$ exceeds $2(n + n')\Lambda$.

Let us examine definitions of \mathbf{W} and \mathbf{V} . Since v_q and w_q vanishes for $|q| > \Lambda$ every individual operator \mathbf{W}, \mathbf{V} changes the total momentum by no more than Λ . Thus, momentum change produced by $(\mathbf{W}^n \mathbf{V}^{n'})$ is no more than $(n + n')\Lambda$. We will use this information in our next step.

First step in evaluating the matrix element from (154) is creation of a quasiparticle-quasihole pair by applying $\tilde{\psi}^\dagger \tilde{\psi}$ product to $|0\rangle$. Since the total momentum of all excitations is zero the momentum of this pair cancels the momentum of $(\mathbf{W}^n \mathbf{V}^{n'})$. Thus, the pair momentum magnitude is limited by $(n + n')\Lambda$. Two elementary excitations, quasiparticle and quasihole, of which the pair is composed have their momenta bound by $(n + n')\Lambda$.

As we explained above, by acting on the ground state the product $\tilde{\psi}^\dagger \tilde{\psi}$ creates an excited state with a quasiparticle and a quasihole. On this state the operator $(\mathbf{W}^n \mathbf{V}^{n'})$ acts creating yet another excited state. Every individual operator \mathbf{V} or \mathbf{W} when acting on a state with finite number of the elementary excitations can either (i) create another quasiparticle-quasihole pair or (ii) replace an elementary excitation with another one of the same chirality and charge but different momentum.

In case (i) both newly created elementary excitations have their momenta bound by Λ . This is because single \mathbf{W}

or \mathbf{V} operator cannot make quasiparticle-quasihole excitation with total momentum higher than Λ .

In case (ii) an individual operator \mathbf{W} or \mathbf{V} can change momentum of an elementary excitation by no more than Λ . As above, this is a consequence of v_q and w_q vanishing at $|q| > \Lambda$. Therefore, the product $(\mathbf{W}^n \mathbf{V}^{n'})$ cannot change momentum of an elementary excitation by more than $(n + n')\Lambda$. Consequently, momentum of an elementary excitation is bound by $2(n + n')\Lambda$.

This means that in (154) summation is effectively performed over a sphere $|k_i| < 2(n + n')\Lambda$, $|p_i| < 2(n + n')\Lambda$. Thus, this expression has no ultraviolet divergence.

The above argument can be easily generalized to other perturbation theory formulas to prove that the perturbation theory for the quasiparticles has no ultraviolet divergences.

The absence of the ultraviolet divergences, together with irrelevance of the quasiparticle - quasiparticle interaction, implies that the quasiparticle perturbation theory is well-defined.

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